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Mathematics News Letter

SUBSCRIPTION $1.00 PER YEAR Single Copies, 15c

Published under auspices of the Louisiana-Mississippi Section of the Mathematical Association of America and the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics.

To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 5 BATON ROUGE, LA., JANUARY, 1931 NO. 5

THE ANNUAL LOUISIANA-MISSISSIPPI MATHEMATICAL MEET.

On another page is a detailed announcement by Chairman Nichols of the regular annual joint meetings of the Louisiana-Mississippi Section of the Mathematical Association of America and the Louisiana-Mississippi Branch of the National Council of the Teachers of Mathematics. In the same announcement is described the main program of the Louisiana Academy of Science whose annual meeting also has a joint relation to that of the Council and the Section, a relation which was approved by vote of the three bodies concerned several years ago.

Those features of the triple program which are published in this issue of the News Letter indicate that an unusually worthwhile and effective program is in store for those who are to be at Natchitoches on the 13th and 14th of this month.

The time has about passed when the serious mathematical worker can afford to be indifferent to the opportunities for contact with his fellow workers which are offered by such conventions as the one about to be held in Natchitoches. The proposition may be put still more strongly. The mathematical teacher who is eager for the highest levels of efficiency will even undergo a proper degree of sacrifice in order to profit professionally by such a convention.

A critically important phase in the work of advancing the

cause of mathematics and its teaching in our two states has been reached, with the result that one may easily forecast that the Natchitoches meeting will be the most important Section-Council meeting that will have been held since the initial joint meeting in Shreveport some years ago.

All those college or secondary teachers who have our cause vitally at heart should go to Natchitoches next Friday. —*S. T. S.*

FROM THE MATHEMATICAL ASSOCIATION OF AMERICA

Oberlin, Ohio, February 24, 1931.

To Teachers of Mathematics
in Louisiana and Mississippi,
Colleagues and Friends:

We feel that you are conducting an experiment which is unique in promoting cooperation between the Mathematical Association of America and the National Council of Teachers of Mathematics. There are many problems in the teaching of mathematics which can best be solved by such a cooperation.

We urge you, therefore, one and all to identify yourselves with one or the other of these two bodies, and especially to support your own **Mathematics News Letter**, which is the only publication of its kind in this country.

Your subscription to the **News Letter** is needed to enable it to continue its good work, and your contributions to its columns will help to form a clearing house for a mutual exchange of ideas that is bound to result in betterment for the cause of mathematics within your sphere of influence. The annual subscription of one dollar should be sent to Professor S. T. Sanders, Louisiana State University, Baton Rouge, La.

The Mathematical Association of America sends you greetings for the New Year with the hope that 1931 may prove the best of all for the promotion of the interest of mathematics. The officers of the Association will hope to see you at the science meetings at New Orleans next December, at which time the Mathematical Association as well as the American Mathematical Society will meet under the auspices of the American Association for the Advancement of Science. You will be able to meet many of the prominent mathematicians of the country and to hear attractive papers on a great variety of mathematical topics.

Yours very sincerely,

W. D. CAIRNS, Secretary.

A NOTE FROM PROFESSOR W. H. TAYLOR ON BAGLEY'S VIEW OF MATHEMATICAL DISCIPLINES.

We are pleased to be able to print the following, bearing as it does, so directly upon the matter of mental discipline.

ALABAMA COLLEGE,
Department of Mathematics
Montevallo, Ala., February 23, 1931.

Prof. S. T. Sanders, Editor and Manager,
Mathematics News Letter,
Baton Rouge, La.

My dear Professor Sanders:

It is with pleasure that I acknowledge receipt, through the kind offices of one of my students, of the December number of the Mathematics News Letter. I find it an interesting and stimulating publication.

In the article by C. D. Smith, I note that a quotation from Paul Shorey seems to imply that Professor Bagley is to be classified among the non-disciplinarians. I am reminded of the time, not so many years ago—nor yet so few—it was my privilege to be the only student registered for one of Professor Bagley's courses. Having at that time recently escaped from the tutelage of a disciple of the Dewey-Thorndike school, I pressed Professor Bagley rather closely on the question of disciplinary values. Much to my gratification, he exhibited the common sense attitude that a change from the Faculty Psychology to the Neuronic one could not affect a phenomenon, but could at most affect the language in terms of which the phenomenon is described. In brief, at that time Professor Bagley was decidely an exponent of the disciplinary idea, *properly interpreted.*

I am inclined to suspect that his attitude has not changed with the years.

Fraternally yours,
WARD H. TAYLOR,
Head Dept. of Math.

"INFINITY"

Recently a new trigonometry text has been published in the preface of which it is said that the text conforms to a recent current tendency, namely, the tendency of writers to avoid such statements as that "the tangent of $90^\circ = \infty$ ", or $a/0 = \infty$ "

We heartily hail the coming of such a text. More than this,

we are delighted at the signs that such a "tendency" has at last arrived. It should have arrived long ago. There is absolutely no sound reason for the mystic statement that if we divide 3 by zero it will give "infinity". We are convinced that such a usage is no more than a relic of the time when the element of mysticism was regarded as an appropriate part of every mathematical essay, when every proposer of a difficult problem was entitled to preserve his own solution as an occult secret to be revealed only to the properly initiated or upon the communication of a correct pass word.

Surely it is during the first or the second year of the student's mathematical experience that he should be required to begin the building of the habit of crystal-clear formulation of mathematical ideas. This is far from saying that he should be required to formulate them with formal rigor at such an early stage of his studies. Formal rigor of statement and clearness of statement are by no means necessarily the same.

No amount of foot-note explanation that "∞" is a mere short-hand for certain concepts is sufficient to counteract the cumulative vicious influence of the constant use of a form of notation which on its face violates a fundamental of algebra, namely, that division by zero is an invalid operation.

On the other hand let us view the purely scientific use of the ideas associated with "infinity."

When we say that as x is allowed to assume values closer and closer to $\pi/2$, $x<\pi/2$, the tangent of x assumes larger and larger postive values without any limit, we are putting the facts in such a form of statement that no more than these facts are expressed. It is a scientific statement. Distinct from this is the companion statement. As x assumes values closer and closer to $\pi/2$, $x>\pi/2$, the tangent of grows larger without limit passing through negative values. We submit that this form of expression, lengthy as it is, is incomparably better than such a statement as, "tan $\pi/2=\pm\infty$"

But the formulation does not need to be lengthy when written out. We make it practice to have the student write it as follows:

$$\text{As } x \longrightarrow \pi/2,\ x<\pi/2,\ \tan x \longrightarrow +\infty.$$

The actual interpretation of this written symbolism is as above.

The particular function here cited is, of course, only one of indefinitely many which could have been used for illustrative purposes. —*S. T. S.*

ANNOUNCEMENT OF NATCHITOCHES MEETINGS MARCH 13, 14

By CHAIRMAN IRBY C. NICHOLS

The eyes of all teachers of mathematics and the general sciences of the Colleges and High Schools of Louisiana and Mississippi are now turning to Natchitoches, where, on Friday and Saturday, March 13-14, under the auspices of the State Normal College of Louisiana will be held the next joint annual convention of these three prominent organizations: The Louisiana-Mississippi Section of the Mathematical Association of America, The Louisiana-Mississippi Branch of the National Council of Mathematics Teachers, The Academy of Science of Louisiana. The first mentioned organization is the oldest. It was organized at Baton Rouge in 1924, and hence is now preparing to hold its seventh annual convention. Its membership is composed of the teachers of College Mathematics. The second organization was organized at Shreveport in 1927; its membership is composed of teachers of High School Mathematics. The Louisiana Academy of Science was organized at Shreveport in 1927. The membership of the Academy is composed of persons interested in any branch of scientific endeavor.

All of the annual meetings of these organizations have been well attended, and in all of them the interest and spirit of professional skill and cooperation has been well sustained. This year a still bigger and better meeting is expected. The program committee announces that a splendid program is guaranteed, and that the faculty and students and the many local friends of the Louisiana Normal are putting forth every effort to make the convention a success in every detail. Hotel Nakatosh will be headquarters in the matter of hotel accomodations; its published rates for rooms and meals are reasonable, and its convenience and hospitality well known. Delegates to the convention should write for reservations at once. Train and bus schedules should also be studied now. In the afternoon, tea will be served by the ladies of the Natchitoches Chapter of the American Association of University women. A big banquet, as usual, will be one of the features of the Friday evening program; this year the affair will be entirely complimentary to all visiting delegates. The annual Friday evening lecture will be given by one of the Louisiana Academy mem-

bers: Professor Frans Blom, Head of the Department of Middle American Research, Tulane University, will give his illustrated lecture on "The Great Cities of Ancient America". This lecture will be very profitable and, undoubtedly, most interesting. Professor Blom will speak either Friday Afternoon or Saturday morning on "Maya Numbers," most probably Saturday morning before the Council.

The program in general will begin at 1:30 P. M. Friday and close at 12 Noon Saturday. In a few days, a completed program will be announced. The Academy will hold its own separate meetings. The Section will render the major portion of its program Friday afternoon, and the Council Saturday morning, although it is anticipated that the numbers of both organizations will be of mutual interest to the entire membership. The Friday evening lecture will be a joint lecture sponsored by all three organizations concerned.

To date the following have accepted invitations to serve on the program of the Academy:

1. Professor R. F. Clark, Louisiana Polytechnic Institute, "Snakes".

2. Professor Paul T. Jones, Louisiana College, "Conductivity Measurements of Aqueous Solutions".

3. Professor Leo Joseph Lassalle, Louisiana State University, "The Present Status of Our Knowledge as to the Nature and Origin of Cosmic Rays".

4. Professor E. H. Herrick, Louisiana State Normal College, "The Ovary as an organ of Internal Secretion".

5. Professor F. T. Morse, Louisiana Polytechnic Institute, "Some Speculations Regarding Cosmic Structure and the Place of Man Therein".

6. Professor I. Mazlish, Centenary College of Louisiana, "The Elements of Theory of Relativity".

7. Professor John S. Kyser, Louisiana State Normal College, "The Role of Distributaries in the Flood Control Program for the Lower Mississippi".

An incomplete list of the speakers on the Section program includes:

1. Dr. H. E. Buchanan, Tulane, "The Construction of the Helium Atom".

2. Dr. B. E. Mitchell, Millsaps College, "Sextantal Analysis".

3. Frank Atkinson Rickey, Mandeville, Louisiana, "On Certain Partial Derived Functions".

4. Professor Nola L. Anderson, Newcomb, "Trigonometry in a Space of N-Dimensions".

5. Dr. C. D. Smith, A. & M., Mississippi, "The Rise of a New Geometry".

6. Professor C. G. Jaeger, Tulane, "A Certain Theorem Concerning Two Triangles".

7. Professor Perry Cole, Ruston, "The Rhind Mathematical Papyrus".

The tentative program for the Council includes:

1. Professor M. C. Rhodes, University of Mississippi, "How Can Mathematics Be Taught Most Effectively?"

2. Professor Ralph L. O'Quin, Louisiana State University, "High School and College Geometry".

3. Professor J. T. Harwell, Shreveport, "Seeing Things".

4. Professor E. M. Shirley, Ruston, "Mathematics in Secondary Schools".

5. Professor S. T. Sanders, Louisiana State University, "Our National Outlook".

6. Professor Leora Blair, Louisiana State Normal, "Some of the Phases of High School Arithmetic".

7. Professor L. S. Miller, Many, Louisiana, "Investigations Versus the Traditional Method of Studying Plane Geometry".

8. Dr. C. D. Smith, A. & M. Mississippi, "The Place of Algebra in the Mathematics Program".

MAJOR OBJECTIVES IN GEOMETRY

By F. A. RICKEY

(Adapted from the writer's discussion of the subject before the Mathematics Section of the L. T. A. at the 1930 meeting.)

To John C. Stone we owe the following statement of the objectives of geometry, brief but inclusive:

"1. Training in logical thinking.
2. The practical uses of the subject.
3. Preparation for other mathematics.
4. A certain cultural aim."

In the Second Yearbook of the National Council of Mathematics teachers, we find a very suggestive set of geometry aims as submitted by Gertrude E. Allen:

"I. To develop space intuition by:

a. Laying a foundation of experience upon which to build.

1. By experiment and measurement.

2. By constructions.

3. By observation of geometric forms in nature, architecture, and in decorative design.

4. By exercise of spatial imagination.

b. Organizing a body of knowledge out of this experience. The definite goal is:

1. To gain an accurate knowledge of the significant properties of geometry.

2. To develop and learn for practical use the essential formulas of mensuration.

3. To reveal possibilities in further exploration and to find incentives to carry on.

c. Applying the resulting knowledge to practical use in the concrete world.

II. To furnish favorable material for exercise in the process of logical thinking.

a. To develop an understanding and appreciation of the more mature method of deductive reasoning in the field of geometry.

b. To form habits of exact, truthful statement and of logical organization of ideas in this field.

c. To establish and exercise a conscious technique of thinking—using as a basis Professor Dewey's analysis of a complete thought as follows:

1. A felt difficulty.

3. Suggestions of possible solution.

4. Development of reasoning of the bearings of the suggestion.

5. Further observation and experiment leading to its acceptance or rejection.

d. To foster all possible transfer of ability to the solution of new problems both mathematical and non-mathematical.

A short discussion of the subject of geometry objectives may easily become shallow because of its breadth. I therefore inquire, "Which of the above objectives (which are typical of most listings) is the most vital one and how may our course in geometry accomplish it?"

The New York Times once defined an educated man as a man who knows when a thing is proved. Arthur Shultze says, "More than one man has testified that he owes his success in life to the habits of exact thinking which he formed when studying mathematics." J. W. A. Young states that "Still more important than the subject matter of geometry is the fact that it exemplifies

most typically, clearly and simply certain modes of thought which are of the utmost importance to everyone." E. H. Taylor, in the April 1930 issue of the Mathematics Teacher expresses the opinion that "The primary reason for studying geometry in the high school is to teach the methods of demonstrative geometry." I join these and many others in giving first place to the objective "Training in logical thinking." Not only am I convinced that it is the most important aim of geometry, but I am also convinced that we are not at present attaining it with any degree of satisfaction. Not that we can expect to rebuild a mentality with a session's course in geometry, but the point is that we **can** inculcate a habit of logical attack and convincing presentation of solution which will increase the ability for meeting problems in and out of school.

However, let us remember with John Dewey that "Logical arrangement is the goal and not the point of departure . . . what is important is that the mind should be sensitive to problems and skilled in their solution." Geometric demonstration involves the application of formal logic to definitions, postulates, and theorems. We must realize that these fundamental ideas have to be formulated, need illustration with familiar objects and concepts, and are mastered only after long and carefully directed use. The most significant cause of early discouragement and later failure is a lack of adequate introduction to geometric notions through concrete experience and intuition before beginning formal proofs. Pupils who learn and use the congruent triangle theorems long before they can tell us what "the same thing" is in "Things equal to the same thing are equal to each other" find themselves memorizing proofs of theorems and "originals" in an effort to "get by" which usually fails. Others, with less ability to memorize, are soon lost in a maize of strange concepts and forms. More care is usually used by the text and by the teacher in illustrating and applying the theorems than the axioms and postulates. As E. H. Taylor says, "Probably that is because the mere statement of the axiom is supposed to make the truth break out with blazing and clarifying light." I therefore say that the most important single factor in reaching our objective in geometry is an adequate introduction.

Germany recognizes the need of a constructive introduction to the subject and the Government Program of Studies gives lengthy details of the material and methods of the introductory part of the course. School systems in America having the junior

high school organization which offers a course containing a large amount of intuitive geometry are meeting the need. The rest of us will have to formulate our own introduction, building on to the insufficient one found in the adopted textbook. The following aims might be criteria of the effectiveness of our introduction:

"1. It should create interest: by pointing out geometric forms in nature; by showing uses of geometry; by giving some of the history of geometry; by arousing a desire to give proofs.

2. It should teach the meaning of necessary terms.

3. It should give practice in measuring with ruler and compasses.

4. It should contain constructions.

5. It should give familiarity with the facts to be proved later.

6. It should give practice in generalizations—that is, in the discovery of theorems.

7. It should show the need of proof.

8. It should give meaning to postulates.

If teachers who have been hurrying through the introduction in order to get down to "real study" will spend several weeks on an introduction careful planned to reach the objectives just given, they will find that some of the pupils who might otherwise have turned out "dull in geometry," failure material, may possess surprising reasoning power which would have been unable to emerge from a muddle of new ideas not thoroughly understood. A thorough introductory course in intuitive geometry cannot assure logical reasoning, but it gives the pupil the necessary familiarity with the terms and concepts for him to proceed with confidence and insight into the deductive proof later. Intuition gives "hunches" and experiment leads to the true solution. Most scientific and mathematical discoveries have been the result of intuitive foresight.

Another important point in teaching habits of logical thinking in geometry is that there should be no abrupt transition from the introductory intuitional geometry to the systematic, demonstrative geometry. Appeal has been made to the pupil's common sense throughout the introduction. This common sense should not be insulted by requiring proof of certain so called theorems in which the pupil can find no need of nor challenge for proof. If we are striving to give training in logical thinking, we must be sure that there is a felt need for proof on the part of the pupil. In this connection Professor T. P. Nunn proposes a larger list of postulates to include:

1. Equality of vertical angles.
2. Angle properties of parallel lines.
3. Properties of figures not evident from similarity.
4. Properties of figures which can be demonstrated by superposition.

Some of these are rather drastic inroads on the classic list of theorems, but certainly they should set us thinking.

To my mind, there is nowhere a greater need for the recognition of individual differences than exists in the teaching of demonstrative geometry. It is as important not to force the slow pupil through the more intricate proofs as it is not to compel him to prove a theorem for which he sees no need of proof. Out of the great amount of literature written upon the subject of meeting individual needs the zealous teacher of geometry can find information that will give the necessary assistance to meet this problem. I would suggest, in passing, Edward R. Maguire's book "The Group Study Plan of Teaching" as being both inspirational and practical.

But some will ask, "Is this training in logical thinking something which will 'carry over' into other fields? Is it really something which can be used in life outside of the geometry class?" Those who regularly read the Mathematics News Letter have the mathematician's view of the so called disciplinary values of mathematics as expressed editorially by Professor S. T. Sanders. I shall content myself here with quoting in conclusion from the Third Yearbook from an article by David Eugene Smith. He writes:

"If the knowledge of how to arrange a logical proof in geometry can be made of no value to us in other fields in which deductive logic can be applied; if the perfection of geometry does not give us an ideal of perfection that helps us elsewhere in our intellectual life; if the succinctness of expression in the statement of a geometric truth does not set a norm for statements in non-mathematical lines; if the contact with absolute truth does not have its influence upon the souls of us; if the very style of reasoning does not transfer so as to help the jurist, the physician, the salesman, the publicist, and the educator; if the habit of rigorous thinking which usually is first begun in the demonstrative geometry is not a valuable habit elsewhere; if a love for beauty cannot be cultivated in geometry so as to carry over to stimulate a love of beauty in architecture—then let us drop geometry from our required courses."

THE EVOLUTION OF SECURITIES

By DORA M. FORNO,
New Orleans Normal School

Investments and speculation are the keynotes of "big business". Because of the very nature of business, there are certain investments, based upon the needs of any nation of great resources which must be sound, and other investments which are questionable or speculative and are not sound.

Misplaced confidence in the power of the press in advertising extensively investments with profitable interest returns have lead many investors astray. Such misplaced confidence has been for ages ruinous to the fortunes of thousands. Legislators have constantly been enacting laws to safeguard investments in savings-banks, homesteads and trust funds.

A knowledge of investments is essential to the financial welfare of everyone, especially the small investor who cannot afford to risk hard earned savings on speculation. Buying stocks upon a margin has little meaning to the average individual, but, to the small investor who ventures upon speculation, it may mean disaster, or, at any rate, it will need all the mathematical skill of an expert to figure whether the poor investor has sufficient reserve capital to prevent his "margins" from being "wiped out".

Because of this general need for the public to know more of investments, it has been made a part of the course of study in arithmetic in our high schools and in the upper grades of our elementary schools. The students are therefore inducted in a small way into the intricacies of "big business."

In order to arouse greater interest in the study of investments, a brief review of the evolution of securities, both from the historical as well as the financial view-point would be found profitable.

Securities are classified as either governmental or corporate, according to whether they are created by public and political, or private business corporations. Hence, I shall sketch briefly the development of government financing on the one hand and the modern stock corporation on the other.

A study of governmental organizations shows that they are not run for profit, but are more often on the verge of bankruptcy. The ideal state would be that in which the books would balance at the end of the year. With the best efforts of skilled mathematicians and executors, it is hardly feasible for them to calculate just how much taxes must be levied, so as to be able to close the books without surplus or deficit.

Throughout the ages political and economic conditions demanded the spending of large sums of money which depleted the treasury, and the taxes were insufficient to meet the demands. Some new method of finance had to be devised. The simplest method seemed to be the one of "financing with the printing press." It was money that was needed, so why not print all that was necessary. This method of financing proved highly dangerous and futile.

Increasing expensiveness of governments demanded more and more money. Private money lenders were resorted to. In the sixteenth and seventeenth centuries, and even later, kings and queens of Europe borrowed money from money lenders on short term loans, often pledging royal jewels as collateral. These debts were afterwards paid off by royal taxation, refunding, or repudiation. The interest rates were very high, for the money lenders were not always sure of payment since there was great loss to them sometimes through royal repudiation.

The eighteenth century ushered in a new financial policy, that is, interest bearing negotiable certificates, issued by the State and sold to investors who were seeking a safe way to invest their savings. They preferred long-term rather than short-term loans which gave rise to British and French securities without a maturity date. Since 1700 this method of obtaining large funds for government expenses by the sale of interest-bearing government certificates or bonds secured by taxation has continually grown.

The historical background of the stock corporation must be understood before we can appreciate its present economic significance. History reveals the fact that America owes a big debt to corporations. Meeker, the economist says that, "North America has been explored by corporations, colonized by corporations, and developed by corporations." The settlement at Jamestown in 1607, was established by the London Company; the Pilgrim fathers were stockholders of the Plymouth Company chartered

in 1620, each Pilgrim being the holder of one share and having the privilege of buying additional stock at £10 apiece. The Dutch East India Company and the Hudson's Bay Company were corporations that played a big part in the economic development of North America.

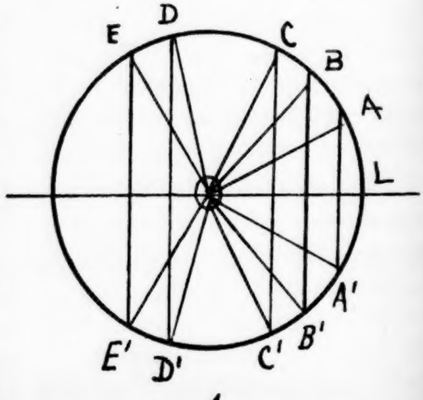
With the development of industries and transportation, has also developed the Stock Exchange and the Trusts, hence a variety of listed securities will be found on the market today. The modern investor must understand the range of possibilities for investment and the advantage of one type of security over another. From the financial standpoint, speculation is mainly responsible for the rapid growth of "big business" in America which is unparalled in history.

INTRODUCTION TO TRIGONOMETRY FROM THE HISTORICAL VIEW POINT

By W. PAUL WEBBER,
Louisiana State University

§1. **Historical.** The Greeks are credited with the invention of trigonometry. The meaning of the term is measurements of three-angled figures, that is triangles. The Greeks made use of the chord of an arc of a circle as the basis of their calculations. Tables of chords were computed for use in solving numerical problems.

It is, of course, obvious that a chord is completely determined by an arc. Hence, the chord may be regarded as a function of the arc. It may be seen in the diagram that each of the arcs A'LA, B'LB, ...E'LE determine the chords A'A, B'B, ...E'E, respectively. But, each arc measures, or determines, an angle at the center. Hence, the chord may also be regarded as a function of the angle. Hipparchus computed the first table of chords for use in calculations.



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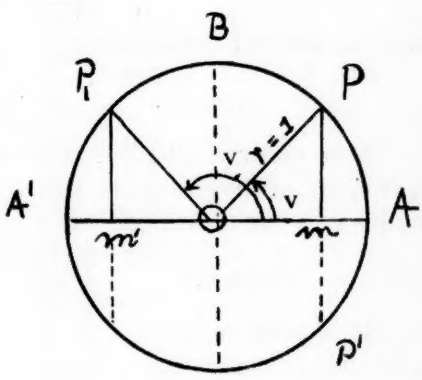
The Greeks, being interested in astronomy, applied their

methods to the solution of triangles on the sphere. Thus, it happened that spherical trigonometry was developed earlier than plane trigonometry which deals with plane triangles.

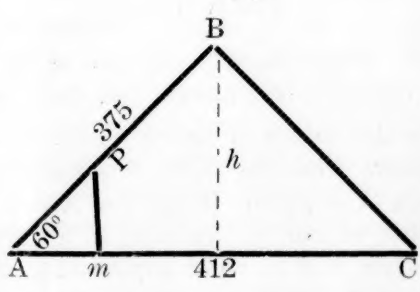
The Hindus, being more interested in land surveying, which is practically a problem of plane triangles developed plane trigonometry further than the Greeks. They modified the method by using the half-chord instead of the chord as the Greeks had done. The computed tables of half-chords of a circle of unit radius later became known as sines of the half angles subtended by the chords at the center.

§2. **Definition of Sine of an Angle.**



The symbol sin v being used for the phrase sine function of angle v, it is easily seen from the diagram that sine v is defined by the above method for all values from v=0° to v=180° This definition obviates the need of making special definitions for sin o°, sin 90°, sin 180°. In the diagram let O P=1, then mP=sin v, m^1P$_1$=sin v$_1$ etc, for all positions of P on A B A'.

2.



By making v$_1$=180—v in the diagram it is easily shown that sin v=sin (180°—v). **Example.** Let it be required to find the altitude and area of the triangle having angle A=60°, side A B=375 and side A C=412.

Construct the triangle and draw a circle of unit radius with center at A. Then mP=sin A. By the similar triangles

$$\frac{mP}{AP} = \frac{h}{AB}$$

whence, h=AB·mP or h=AB sin A, if AP=1

If sin A=sin 60 is taken from a table it is found sin 60=

.866 to 3 decimal places. Hence, on substitution

$$h = 375 \ .866 = 324.+$$

The area of the triangle is

$$\tfrac{1}{2} \cdot 400 \ . \ 324 = 64800.$$

The last equations may be generalized in the form

$$\triangle = \tfrac{1}{2} \text{ base } \cdot \text{altitude}$$
$$= \tfrac{1}{2} \text{ AC } \cdot \text{ AB sin v}$$

which gives the area of any triangle where two sides and their included angle are given.

§3. **Generalization of the example.** From the equation

$$\frac{mP}{AP} = \frac{h}{AB}$$

may be derived the relation

$$mP = \frac{h(AP=1)}{AB} = \frac{h}{AB}$$

Now observe that h is opposite the angle A in the triangle A B h. It follows that the sine function can be defined for acute angles of a right triangle as

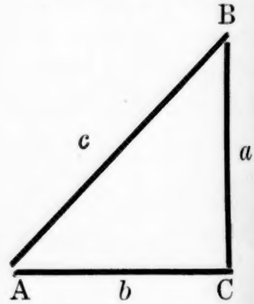
$$\text{Sin A} = \frac{\text{leg opp. A}}{\text{hypotenuse}}$$

This form is current in texts and is useful in solving problems pertaining to right triangles.

Using this definition on the angle A B h=90—A gives

$$\text{Sin A B } h = \frac{\text{leg adj A}}{\text{hypotenuse}} = \frac{\text{leg opp. B}}{\text{hypotenuse}}$$

Since B=90—A, sin A B h is called the **cosine** of A (cos A). This shows that for acute angles the sine of an angle is the cosine of its complement. It appears that cos A is determined when sin A is given. Hence cos A is a function of sin A. Further, in a right triangle A B C, right-angled at C, the ratios of all the sides taken two and two are fixed when A is known. That is, a/c, b/c, a/b, are all known when A is determined.



§4. **Four New Functions Defined.** The ratios

$$\text{tangent of angle A} = \tan \text{A} = \frac{\text{leg opp. A}}{\text{leg adj. A}} = \frac{a}{b}$$

$$\text{cotangent of angle A} = \text{ctn A} = \frac{\text{leg adj. A}}{\text{leg opp. A}} = \frac{b}{a}$$

$$\text{secant of angle A} = \text{sec A} = \frac{\text{hypotenuse}}{\text{leg adj. A}} = \frac{c}{b}$$

$$\text{cosecant of angle A} = \text{csc A} = \frac{\text{hypotenuse}}{\text{leg opp. A}} = \frac{c}{a}$$

are defined for conveninence. They are all determined when A is known in the right triangle.

It may be remarked that any one of the six functions may be defined first and all the others follow as conveniences. Historically the sine was defined and used first. Hence, it is a natural starting point for defining all.

At this point the discussion can be easily hooked on to the treatment in nearly any text.

SEXTANTAL TRIGONOMETRY

By B. E. MITCHELL,
Millsaps College.

Introduction.

Recently* we were considering the geometry of the sextantal triangle, ABC, $C=60^\circ$, $a^2+b^2-ab=c^2$ together with the bisextantal triangle, $C=120^\circ$, $a^2+b^2+ab=c^2$. We proceed now to recast our problem in a larger mold. This we do by replacing a, b, c by x, y, r, r being opposite the naming angle. If, on the one hand, we consider x, y as the (rectangular) cartesian coordinates of a point in the plane, the equation $x^2+y^2-xy=r^2$ will be that of an ellipse with its center at the origin and its major and minor axes making angles of 45° and 135° respectively with the x-axis. If, on the other hand, we consider the axes not as rectangular but as oblique, to be exact, making an angle of 120° with each other, the equation $x^2+y^2-xy=r^2$ will be that of a circle of radius r with center at the origin. It is thus that we shall consider it in this paper. Under this larger aspect of our problem we observe that our two types of triangles, the sextantal and the bisextantal have coalesced, for the sign of the product term wherein they differed is now taken care of by the convention of signs of the coordinates, x and y.

*Mathematics News Letter Vol. 5, Nos. 1, 2.)

Definitions

Previously we borrowed the usual quadrantal trigonometry to discuss our problem. This, however, is not necessary. Let the point P (x, y) describe the circle with center at the origin, 0, and radius r, starting at the positive extremity of the horizontial diameter. Thus the radius OP generates an angle $OAP = u$, say. The six ratios of the three sides of the sextantal (or bisextantal) triangle give us six functions of the angle u. Their symbols, definitions, and values for some of the cardinal angles are given in the following table:

Symbol	Definition	Values 0	30°	60°	90°	120°	180°
$s(u)$	y/r	0	$2/\sqrt{3}$	1	$\sqrt{3}/2$	1	0
$c(u)$	x/r	1	$1/\sqrt{3}$	1	1/2	0	-1
$t(u)$	y/x	0	2	1	$\sqrt{3}$	∞	0
$rt(u)$	x/y	∞	1/2	1	$1/\sqrt{3}$	0	∞
$rc(u)$	r/x	1	$\sqrt{3}$	1	2	∞	-1
$rs(u)$	r/y	∞	$\sqrt{3}/2$	1	$2/\sqrt{3}$	1	∞

We retain the letters s, c t as suggestive of their analogues in quadrantal trigonometry, namely, 'sine', 'cosine', and 'tangent'. We replace the prefix 'co-' of quadrantal trigonometry by 'r' indicating 'reciprocal' since 'complementary angles' do not play a conspicuous role in sextantal trigonometry.

The fact that when $u = 60^\circ$, $x = y = r$ and their several ratios all have the value 1, leads us to adopt 60° as our unit angle and to express all other angles, as occasion may arise, as multiples and submultiples of it. Thus $60^\circ = 1$ sextant or simply 1, $120^\circ = 2$, $30^\circ = 1/2$. etc.

The graphs of these several functions are gotten from those of quadrantal trigonometry by a shearing of the plane as indicated in the first section of this paper. If we carry the word 'graph' back to its Greek equivalent meaning 'to write', then as relates to the curves associated with its functions, guadrantal trigonometry is 'vertical handwriting' while sextantal is (oblique) 'Spencerian'.

Sextantal Equivalents

The analogues of the familiar quadrantal relations are exhibited in the following tables. Let the angle u be less than 60° or 1. This restriction is not necessary, but for the sake of simplicity we shall consider it so for the present.

Table A.

Angles	u	$1-u$	$2-u$	$3-u$	$4-u$	$5-u$	$-u$

Elements	x	x	y	$y-x$	$-x$	$-y$	$x-y$
	y	$x-y$	x	y	$y-x$	$-x$	$-y$
	r	r	r	r	r	r	r
Functions	$c(u)$	$c(u)$	$s(u)$	$s(u)-c(u)$	$-c(u)$	$-s(u)$	$c(u)-s(u)$
	$s(u)$	$c(u)-s(u)$	$c(u)$	$s(u)$	$s(u)-c(u)$	$-c(u)$	$-s(u)$

These relations are readily derivable from figures or from the addition theorems of the next section.

Using the table we sort out the following equivalents:

$$s(u)=c(2-u)=s(3-u)=-c(5-u)=-s(-u)$$
$$c(u)=c(1-u)=s(2-u)=-c(4-u)=-s(5-u)$$
$$c(u)-s(u)=c(-u)=s(1-u)=-c(3-u)=-s(4-u).$$

Angles	u	$1+u$	$2+u$	$3+u$	$4+u$	$5+u$
Elements	x	$x-y$	$-y$	$-x$	$y-x$	y
	y	x	$x-y$	$-y$	$-x$	$y-x$
	r	r	r	r	r	r
Functions	$s(u)$	$c(u)$	$c(u)-s(u)$	$-s(u)$	$-c(u)$	$s(u)-c(u)$
	$c(u)$	$c(u)-s(u)$	$-s(u)$	$-c(u)$	$s(u)-c(u)$	$s(u)$

Sorting again as in Table A.

$$s(u)=-c(2+u)=-s(3+u)=c(5+u)$$
$$c(u)=s(1+u)=-c(3+u)=-s(4+u)$$
$$c(u)-s(u)\quad c(1+u)=s(2+u)=-c(4+u)=-s(5+u)$$

A casual study of the tables might lead one to anticipate new relations in the way of reduction formulas of the nature

$$s(n+u)=f(u) \textit{ and } c(n+u)=g(u)$$

Such formulas in all cases examined by the writer reduce to those of quadrantal trigonometry, which is probably what might be expected.

Addition Theorems.

From figures in every respect analogous to the customary figures in quadrantal trigonometry we derive the following addition theorems:

$$s(u+v)=s(u)c(v)+c(u)s(v)-s(u)s(v)$$
$$s(u-v)=s(u)c(v)-c(u)s(v)$$
$$c(u+v)=c(u)c(v)-s(u)s(v)$$
$$c(u-v)=c(u)c(v)-s(u)s(v)-c(u)s(v)$$

The combination of the first and last of these formulas by addition and the second and third by subtraction leads to interesting results, namely

$$s(u+v)+c(u+v)=[s(u)+c(u)]c(v)$$

and $$c(u+v)-s(u-v)=[c(u)+s(v)]c(u+1)=[s(v)+c(v)]\,s(u+2).$$

The functions assume more symmetrical forms when three angles are used. From the development of $s(u+v+w)$ making use of the tables of the preceding section we deduce the interesting relation:

$$s(3u)=s(u)s(u+1)s(u+2).$$

The fact that $c^3(u)+s^3(u)=[c(u)+s(u)][c^2(u)-c(u)s(u)+s^2(u)]=c(u)+s(u)$ leads us to predict that these functions belong to a larger group based on the relation $x^3+y^3-3xy=$ a constant, or more homogeneously on $x^3+y^3+z^3-3xyz=0$ so rendered by the introduction of a redundant variable z and hence a redundant axis in the plane.

THE METHOD OF SUCCESSIVE SUBSTITUTIONS

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Newton's method of approximating to the roots of an equation is a special case of a more general method known as the method of successive substitutions or the method of iteration. It is the object of this note to give some theorems which justify the method in certain cases.

§1. *The sequence of functions derived from a given function by iteration.* Let $f(x)$ be a function defined for $a\leq x\leq b$ and such moreover that

$$\text{(I)}\quad a\leq f(x)\leq b \quad (a\leq x\leq b).$$

It follows from (I) that the equation

$$f_1(x)=f(f(x))$$

defines a function $f_1(x)$ for $a\leq x\leq b$ which satisfies the condition

$$(\mathrm{I}_1)\quad a\leq f_1(x)\leq b \quad (a\leq x\leq b).$$

It then follows from (I_1) that the equation

$$f_2(x)=f(f_1(x))$$

defines a function $f_2(x)$ for $a\leq x\leq b$ which satisfies the condition

$$(\mathrm{I}_2)\quad a\leq f_2(x)\leq b \quad (a\leq x\leq b).$$

On continuing in this way we obtain an infinite sequence of functions

$$f_0(x)=f(x),\ f_1(x),\ f_2(x),\ f_3(x),\ldots$$

each defined for $a\leq x\leq b$ and such moreover that

$$a\leq f_n(x)\leq b,\ f_n(x)=f(f_{n-1}(x))\quad (n=1,2,3,\ldots).$$

We call this sequence *the sequence derived from $f(x)$ by iteration.*

§2. *Solution of the equation $x=f(x)$ by iteration.* We shall prove

Theorem 1. If $f(x)$ is defined for $a \leqq x \leqq b$ and satisfies condition (I), and if moreover $f(x)$ is continuous for $a \leqq x \leqq b$ and the sequence derived from $f(x)$ by iteration converges to a limit function $F(x)$, then

$$F(x) = f(F(x)).$$

Proof. By definition of $f_n(x)$,

$$f_n(x) = f(f_{n-1}(x)).$$

On passing to the limit in this equation an n approaches infinity, we get the desired result.

It follows from the theorem that every value of $F(x)$ is a solution of the equation $x = f(x)$.

We next state and prove a more precise theorem. To this end we lay down the following definition.

A function $f(x)$ has property (P) if it is defined for $a \leqq x \leqq b$ and is such that there exists a number r such that

$$a \leqq r \leqq b, \quad [f(x) - r][f(x) - x] \leqq 0 \quad (a \leqq x \leqq b).$$

Theorem 2. If $f(x)$ has property (P), then it has property (I) and the sequence derived from it by iteration converges to a function $F(x)$ which satisfies (I). Moreover if $f(x)$ is continuous for $a \leqq x \leqq b$, then

$$F(x) = f(F(x)).$$

Proof. From the definition of property (P) we have the existence of r such that

$$(1) \quad a \leqq x \leqq b, \quad (2) \quad [f(x) - r][f(x) - x] \leqq 0, \quad (a \leqq x \leqq b).$$

By (2) we have

$$(3) \quad x \leqq (\geqq) f(x) \leqq (\geqq) r \quad (a \leqq x \leqq b),$$

where it is understood the inequality signs in the parentheses both hold or else the signs just before those parentheses both hold. But from (1), (3) we get

$$(4) \qquad a \leqq f(x) \leqq b,$$

which is (I).

From (2), (3) we get

$$(5) \quad [f_1(x) - r][f_1(x) - f(x)] \leqq 0 \quad (a \leqq x \leqq b),$$

on replacing x by $f(x)$. Also as above

$$(6) \qquad a \leqq f_1(x) \leqq b.$$

On continuing in this way we secure an infinite sequence of inequalities

$$(7) \quad \begin{aligned} &[f_2(x) - r][f_2(x) - f_1(x)] \leqq 0 \\ &[f_3(x) - r][f_3(x) - f_2(x)] \leqq 0 \\ &[f_4(x) - r][f_4(x) - f_3(x)] \leqq 0 \\ &[f_5(x) - r][f_5(x) - f_4(x)] \leqq 0 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

But (5), (7) imply that for a given value of x, the sequence of values $f(x)$, $f_1(x)$, $f_2(x)$,... is a bounded, monotonic one and hence has a limit, say F (x). This proves the first part of the theorem. The rest then follows from Theorem 1.

§3. *Solution of* $g(x)=0$ *by iteration.* We note that if $f(x)$ is defined by the equation

$$\text{(A)}\quad f(x)=x-g(x)/\mathrm{M}(x),$$

then the equations $g(x)=0$ and $x=f(x)$ are equivalent so long as $\mathrm{M}(x)$ not$=0$. Thus to solve the equation $g(x)=0$ we have only to solve the equation $x=f(x)$ and in doing the latter we have at our disposal the choice of $\mathrm{M}(x)$. There is considerable latitude in this, as the theorems about to be given show.

The statement of the next theorem is facilitated by introducing the function of two variables defined by the following equation

$$g(x_0x_1)=[g(x_0)-g(x_1)]/(x_0-x_1)\qquad (x_0 \text{ not}=x_1).$$

We shall now prove

Theorem 3. If $g(x)$, $\mathrm{M}(x)$ are functions defined for $a\leq x\leq b$, and r is a number between a and b such that

$$\text{(B)}\quad g(r)=0,$$
$$\text{(C)}\quad \mathrm{M}(x) \text{ not}=0\quad (a\leq x\leq b),$$
$$\text{(D)}\quad 0\leq g(xr)/\mathrm{M}(x)\leq 1\quad (a\leq x\leq b),$$

and if $f(x)$ is defined by (A), then $f(x)$ satisfies (I) and the sequence of functions derived from it by iteration converges to a function $\mathrm{F}(x)$ which satisfies (I). If moreover $g(x)$ is continuous for $a\leq x\leq b$ and if $\mathrm{M}(x)$ is continuous for all those values of x except those at which $g(x)=0$ and is bounded at all the latter, then

$$g(\mathrm{F}(x))=0.$$

Proof. The theorem will follow from Theorem 2 if we can show that

$$[f(x)-r][f(x)-x]\leq 0\qquad (a\leq x\leq b).$$

Now

$$g(x)=g(x)-g(r)=g(xr)(x-r)\qquad (x \text{ not}=r).$$

Hence

$$[f(x)-r][f(x)-x]$$
$$=[x-r-g(x)/\mathrm{M}(x)][-g(x)/\mathrm{M}(x)]$$
$$=-(x-r)^2[1-g(xr)/\mathrm{M}(x)][g(xr)/\mathrm{M}(x)]$$
$$\leq 0\qquad (a\leq x\leq b)\ (x \text{ not}=r)$$

But

$$[f(x)-\mathrm{r}][f(x)-\mathrm{x}]=0\qquad (x=r).$$

The proof is thus complete.

Theorem 4. If $f(x)$, $M(x)$ are defined for $a \leq x \leq b$ $g(x)$ is continuous for these values, and

$$\text{(E)} \quad g(a)g(b) < 0,$$

$$\text{(F)} \quad 0 \leq g(x_0 x_1) \leq 1 \quad (a \leq x \leq b, a \leq x_0 < x_1 \leq b, \; x_0 \leq x_1),$$

and if $f(x)$ is defined by (A), then $f(x)$ satisfies (I) and the sequence of functions derived from it by iteration converges to a function $F(x)$ which satisfies (I). If moreover $M(x)$ is continuous for all values of x such that $a \leq x \leq b$ except those at which $g(x)=0$ and is bounded at the latter, then $g(F(x))=0$.

Proof. By (E) there exists a number r between a and b such that $g(r)=0$. Then (F) implies $0 \leq g(xr)/M(x) \leq 1 \quad (a \leq x \leq b)$. The theorem then follows from Theorem 3.

We now introduce the function of three variables defined by the equation

$$g(x_0 x_1 x_2) = [g(x_0 x_1) - g(x_0 x_2)]/(x_1 - x_2).$$

Here x_0, x_1, x_2 are understood to be distinct.

Theorem 5. If $g(x)$ is defined and has a continuous first derivative $g^1(x)$ for $a \leq x \leq b$ and

$$\text{(E)} \quad g(a)g(b) < 0,$$

$$\text{(G)} \quad g^1(x) \text{ not} = 0 \quad (a \leq x \leq b),$$

$$\text{(H)} \quad g(x_0 x_1 x_2) \text{ not} = 0 \quad (a \leq x_0 < x_1 < x_2 \leq b),$$

and if $G = g(a, \tfrac{1}{2}(a+b), b)$ and c is a number and $M(x)$ a function such that

$$\text{(J)} \quad a \leq c \leq b, \qquad Gg(c) \leq 0,$$

$$\text{(K)} \quad \begin{aligned} M(x) &= g_1(x) \quad \text{if} \quad Gg(x) \leq 0, \\ &= g\,(x) \quad \text{if} \quad Gg(x) < 0, \end{aligned}$$

and if $f(x)$ is defined by (A), then $f(x)$ is continuous and satisfiies condition (I). Moreover the sequence of functions derived from $f(x)$ by iteration converges to a function $F(x)$ which satisfies (I) and is such that $g(F(x))=0$.

Proof. We note first that (H) implies that G and $g(x_0,x_1,x_2$ have the same sign for all x_0,x_1,x_2 such that $a \leq x_0 < x_1 < x_2 \leq b$. We also note that (E), (G) imply that there is a uniquely determined number r such that $a < r < b$ and $g(r)=0$.

Now suppose that $Gg(x) \leq 0$. We have

$$g(xr) = [g(x) - g(r)]/(x-r) = g(x)/(x-r).$$

Hence

$$\text{(1)} \quad x - r = g(x)/g(xr).$$

Also

$$g(xx_1) - g(xr) = g(xx_1 r)(x_1 - r).$$

On taking the limit as x_1 approaches x, we get

$$(2)\quad g^1(x)-g(xr)=g(xxr)(x-r),$$

where $g(xxr)$ denotes the limit of $g(xx_1r)$ as x_1 approaches x. It also follows from (H) that $g(xxr)$ has the same sign as G.

From (1), (2)

$$g^1(x)-g(xr)=[g(xxr)g(x)]/g(xr)$$

so that

$$(3)\quad 1-g(xr)/g^1(x)=[g(xxr)g(x)]/[g(xr)g^1(x)]$$

It follows from the mean value theorem and (G) that $g(xr)g^1(x)$ is positive. From this and (3)

$$(4)\quad 0<g(xr)/g^1(x)<1 \qquad (Gg(x)\leq 0)$$

Now suppose $Gg(x)<0$. By (4)

$$(5)\quad 0<g(cr)/g^1(c)<1$$

Also

$$g(cx)=[g(c)-g(x)]/(c-x).$$

Hence

$$(6)\quad c-x=[g(c)-g(x)]/g(cx).$$

But

$$g(cr)-g(xr)=g(xcr)(c-x)$$

which gives on using (6)

$$g(cr)-g(xr)=[g(c)g(xcr)-g(x)g(xcr)]/g(cx),$$

or

$$1-g(xr)/g(cr)=[g(c)g(xcr)-g(x)g(xcr)]/[g(cx)g(cr)].$$

But

$$g(c)g(xcr)\leq 0,\quad g(x)g(xcr)<0,\quad g(cr)g(cr)>0.$$

Hence

$$(7)\quad 0<g(xr)/g(cr)<1 \qquad (Gg(x)<0).$$

But (5), (7) give

$$(8)\quad 0<g(xr)/g^1(c)<1 \qquad (Gg(x)<0).$$

Finally from (4), (8) we get

$$0<g(xr)/M(x)<1 \qquad (a\leq x\leq b).$$

The theorem now follows from Theorem 3.

Corollary. Theorem 5 remains true if (H) is replaced by

$$(H')\quad g^{11}(x) \text{ exists not}=0 \qquad (a\leq x\leq b),$$

and G is defined as $g^{11}(a)$.

For we have

$$(9)\quad g(x_0x_1x_2)=\tfrac{1}{2}g^{11}(X),$$

where X is between the smallest and the largest of x_0, x_1, x_2. But (9) shows that (H′) implies (H). Moreover it is easily shown that $g^{11}(a)g(a, \tfrac{1}{2}(a+b), b)>0$.